USA Mathematical Talent Search

Round 3 Problems

Year 33 — Academic Year 2021-2022

`www.usamts.org`

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name, username, and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.
4. Submit your solutions by **January 4, 2022** via one (and only one!) of the methods below:
   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on January 4, 2022.
   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before January 4, 2022.
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to `www.usamts.org` and visiting the "My USAMTS" pages.
7. Round 3 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

USA Mathematical Talent Search

Round 3 Problems

Year 33 — Academic Year 2021-2022

`www.usamts.org`

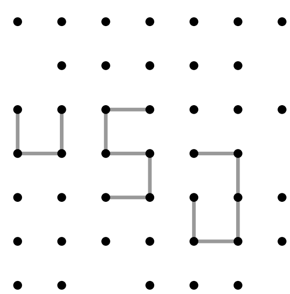
Each problem is worth 5 points.

1/3/33. In the grid below, draw horizontal and vertical segments of unit length joining pairs of adjacent dots (some have been given to you) so that

1. every dot is connected by line segments to exactly 1 or 3 adjacent dots,
2. any dot can be reached from any other dot by following a path of segments, and
3. no area is completely enclosed by segments.

Note: "Unit length" is the length between two adjacent dots when there is no missing dot between them. For example, we cannot draw a vertical line segment down from the dot in the top right corner because the length of this segment would be 2 units.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



2/3/33. Sydney the squirrel is at $(0, 0)$ and is trying to get to $(2021, 2022)$. She can move only by reflecting her position over any line that can be formed by connecting two lattice points, provided that the reflection puts her on another lattice point. Is it possible for Sydney to reach $(2021, 2022)$?

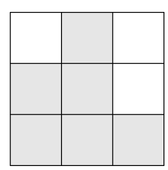
Lattice points are points in the Cartesian plane where both coordinates are integers.

3/3/33. Let n be a positive integer. Let S be the set of n^2 cells in an $n \times n$ grid. Call a subset T of S a **double staircase** if

1. T can be partitioned into n horizontal nonoverlapping rectangles of dimensions 1×1, 1×2, ..., $1 \times n$, and
2. T can also be partitioned into n vertical nonoverlapping rectangles of dimensions 1×1, 2×1, ..., $n \times 1$.

In terms of n, how many double staircases are there? (Rotations and reflections are considered distinct.)

An example of a double staircase when $n = 3$ is shown below.



4/3/33. Let ABC be a triangle whose vertices are inside a circle Ω. Prove that we can choose two of the vertices of ABC such that there are infinitely many circles ω that satisfy the following properties:

1. ω is inside of Ω,
2. ω passes through the two chosen vertices, and
3. the third vertex is in the interior of ω.

5/3/33. Let a, b, c, d be positive real numbers. Prove that d is an integer **if and only if** there are positive real numbers e, f satisfying

$$\left\lfloor \frac{\left\lfloor \frac{x+a}{b} \right\rfloor + c}{d} \right\rfloor = \left\lfloor \frac{x+e}{f} \right\rfloor$$

for all real numbers x. (For a real y, $\lfloor y \rfloor$ is the greatest integer less than or equal to y.)

Problems by Thomas Lam, Kevin Ren, and USAMTS Staff.
Round 3 Solutions must be submitted by **January 4, 2022**.
Please visit **http://www.usamts.org** for details about solution submission.
© 2021 Art of Problem Solving Initiative, Inc.