USA Mathematical Talent Search

Round 2 Problems

Year 32 — Academic Year 2020-2021

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **November 30, 2020** via one (and only one!) of the methods below:

   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on November 30, 2020.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before November 30, 2020.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to `www.usamts.org` and visiting the "My USAMTS" pages.

7. Round 2 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

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Each problem is worth 5 points.

1/2/32. In the grid below, fill each gray cell with one of the numbers from the provided bank, with each number used once, and fill each white cell with a positive one-digit number. The number in a gray cell must equal the sum of the numbers in all touching *white* cells, where two cells sharing a vertex are considered touching. All of the terms in each of these sums must be distinct, meaning that two white cells with the same digit may not touch the same gray cell.

	5		(gray)		8	
4	(gray)				(gray)	
	3		(gray)		(gray)	1
	(gray)				7	
	8		(gray)	1	(gray)	4
	(gray)				2	

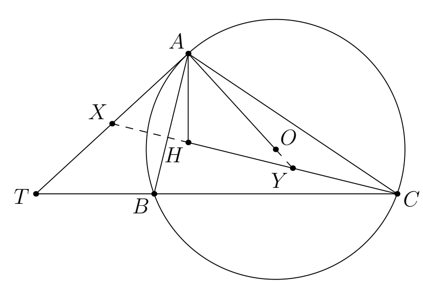
Bank: 15, 23, 28, 35, 36, 38, 40, 42, 44

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: in any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/32. Infinitely many math beasts stand in a line, all six feet apart, wearing masks, and with clean hands. Grogg starts at the front of the line, holding n pieces of candy, $n \geq 1$, and everyone else has none. He passes his candy to the beasts behind him, one piece each to the next n beasts in line. Then, Grogg leaves the line. The other beasts repeat this process: the beast in front, who has k pieces of candy, passes one piece each to the next k beasts in line, and then leaves the line. For some values of n, another beast, besides Grogg, temporarily holds all the candy. For which values of n does this occur?

3/2/32. Given a nonconstant polynomial with real coefficients $f(x)$, let $S(f)$ denote the sum of its roots. Let p and q be nonconstant polynomials with real coefficients such that $S(p) = 7$, $S(q) = 9$, and $S(p - q) = 11$. Find, with proof, all possible values for $S(p + q)$.

4/2/32. Let ABC be a triangle with $AB < AC$. As shown below, T is the point on $\overline{BC}$ such that $\overline{AT}$ is tangent to the circumcircle of $\triangle ABC$. Additionally, H and O are the orthocenter and circumcenter of $\triangle ABC$, respectively. Suppose that $\overline{CH}$ passes through the midpoint of $\overline{AT}$. Prove that $\overline{AO}$ bisects $\overline{CH}$.



5/2/32. Let a_1 be any positive integer. For all i, write 5^{2020} times a_i in base 10, replace each digit with its remainder when divided by 2, read off the result in binary, and call that a_{i+1}. Prove that $a_N = a_{N+2^{2020}}$ for all sufficiently large N.

Problems by David Altizio, Nikolai Beluhov, Daniel Ghenghea, Michael Tang, and USAMTS Staff.
Round 2 Solutions must be submitted by **November 30, 2020**.
Please visit **http://www.usamts.org** for details about solution submission.