USA Mathematical Talent Search

Round 2 Problems

Year 29 — Academic Year 2017–2018

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your username and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by November 27, 2017, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on November 27, 2017

   (b) Mail: USAMTS
   P.O. Box 4499
   New York, NY 10163
   (Solutions must be postmarked on or before November 27, 2017.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 2 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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Each problem is worth 5 points.

1/2/29. Given a rectangular grid with some cells containing one letter, we say a row or column is *edible* if it has more than one cell with a letter and all such cells contain the same letter. Given such a grid, the hungry, hungry letter monster repeats the following procedure: he finds all edible rows and all edible columns and simultaneously eats all the letters in those rows and columns, removing those letters from the grid and leaving those cells empty. He continues this until no more edible rows and columns remain. Call a grid a *meal* if the letter monster can eat all of its letters using this procedure.

U		T			S	S
T		T		T	T	
A		A			S	A
A	T	A		M	S	
			M			
		T	U	M		
		A		M		

In the 7 by 7 grid to the right, fill each empty space with one letter so that the grid is a meal and there are a total of eight Us, nine Ss, ten As, eleven Ms, and eleven Ts. Some letters have been given to you.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/29. Let b be a positive integer. Grogg writes down a sequence whose first term is 1. Each term after that is the total number of digits in all the previous terms of the sequence when written in base b. For example, if $b = 3$, the sequence starts $1, 1, 2, 3, 5, 7, 9, 12, \ldots$. If $b = 2521$, what is the first positive power of b that does not appear in the sequence?

3/2/29. The USAMTS tug-of-war team needs to pick a representative to send to the national tug-of-war convention. They don't care who they send, as long as they don't send the weakest person on the team. Their team consists of 20 people, who each pull with a different constant strength. They want to design a tournament, with each round planned ahead of time, which at the end will allow them to pick a valid representative. Each round of the tournament is a 10-on-10 tug-of-war match. A round may end in one side winning, or in a tie if the strengths of each side are matched. Show that they can choose a representative using a tournament with 10 rounds.

Problems by Nikolai Beluhov, Billy Swartworth, Michael Tang, and USAMTS Staff.
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4/2/29. Zan starts with a rational number $0 < \frac{a}{b} < 1$ written on the board in lowest terms. Then, every second, Zan adds 1 to both the numerator and denominator of the latest fraction and writes the result in lowest terms. Zan stops as soon as he writes a fraction of the form $\frac{n}{n+1}$, for some positive integer n. If $\frac{a}{b}$ started in that form, Zan does nothing.

As an example, if Zan starts with $\frac{13}{19}$, then after one second he writes $\frac{14}{20} = \frac{7}{10}$, then after two seconds $\frac{8}{11}$, then $\frac{9}{12} = \frac{3}{4}$, at which point he stops.

(a) Prove that Zan will stop in less than $b - a$ seconds.

(b) Show that if $\frac{n}{n+1}$ is the final number, then

$$\frac{n-1}{n} < \frac{a}{b} \leq \frac{n}{n+1}.$$

5/2/29. There are n distinct points in the plane, no three of which are collinear. Suppose that A and B are two of these points. We say that segment AB is *independent* if there is a straight line such that points A and B are on one side of the line, and the other $n-2$ points are on the other side. What is the maximum possible number of independent segments?

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