USA Mathematical Talent Search

Round 2 Problems

Year 25 — Academic Year 2013–2014

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by November 18, 2013, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on November 18

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before November 18.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 2 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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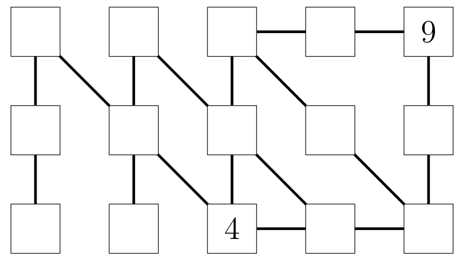
Each problem is worth 5 points.

1/2/25. In the 3×5 grid shown, fill in each empty box with a two-digit positive integer so that:



(a) no number appears in more than one box, and

(b) for each of the 9 lines in the grid consisting of three boxes connected by line segments, the box in the middle of the line contains the least common multiple of the numbers in the other two boxes on the line.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/25. Let $ABCD$ be a quadrilateral with $\overline{AB} \parallel \overline{CD}$, $AB = 16$, $CD = 12$, and $BC < AD$. A circle with diameter 12 is inside of $ABCD$ and tangent to all four sides. Find BC.

3/2/25. For each positive integer $n \geq 2$, find a polynomial $P_n(x)$ with rational coefficients such that $P_n\left(\sqrt[n]{2}\right) = \dfrac{1}{1+\sqrt[n]{2}}$. (Note that $\sqrt[n]{2}$ denotes the positive n^{th} root of 2.)

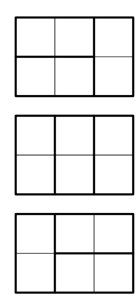
4/2/25. An infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ is called *spooky* if $a_1 = 1$ and for all integers $n > 1$,

$$na_1 + (n-1)a_2 + (n-2)a_3 + \cdots + 2a_{n-1} + a_n < 0,$$
$$n^2a_1 + (n-1)^2a_2 + (n-2)^2a_3 + \cdots + 2^2a_{n-1} + a_n > 0.$$

Given any spooky sequence $a_1, a_2, a_3, \ldots$, prove that

$$2013^3a_1 + 2012^3a_2 + 2011^3a_3 + \cdots + 2^3a_{2012} + a_{2013} < 12345.$$

5/2/25. Let S be a planar region. A *domino-tiling* of S is a partition of S into 1×2 rectangles. (For example, a 2×3 rectangle has exactly 3 domino-tilings, as shown to the right.) The rectangles in the partition of S are called *dominoes*.



(a) For any given positive integer n, find a region S_n with area at most $2n$ that has exactly n domino-tilings.

(b) Find a region T with area less than 50000 that has exactly 100002013 domino-tilings.

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Please visit **http://www.usamts.org** for details about solution submission.